The Principle of Symmetric Criticality for Non-Differentiable Mappings

Technical Report No.2003-1

Jun Kobayashi
Department of Applied Physics, School of Science and Engineering, Waseda University

Mitsuharu Ōtani
Associated Researcher of Advanced Research Institute for Science and Engineering, Waseda University
Department of Applied Physics, School of Science and Engineering, Waseda University

April, 2003

ADVANCED RESEARCH INSTITUTE FOR SCIENCE AND ENGINEERING, WASEDA UNIVERSITY

The primary intent of this publication, Technical Report, is to share original work as quickly as possible. Therefore, articles which appear are not reviewed as is the usual practice with most journals. The authors alone are responsible for the content, interpretation of data, and opinions expressed in the articles. All communications concerning the articles should be addressed to the author c/o:
The Principle of Symmetric Criticality for Non-Differentiable Mappings

Jun Kobayashi *
Department of Applied Physics, School of Science and Engineering, Waseda University

Mitsuharu Otani †
Associated Researcher of Advanced Research Institute for Science and Engineering, Waseda University
Department of Applied Physics, School of Science and Engineering, Waseda University
e-mail: kobajun@aoni.waseda.jp; otani@waseda.jp

April, 2003

Abstract
Let $X$ be a Banach space on which a symmetry group $G$ linearly acts and let $J$ be a $G$-invariant functional defined on $X$. In 1979, R. Palais [17] gave some sufficient conditions to guarantee the so-called “Principle of Symmetric Criticality”: every critical point of $J$ restricted on the subspace of $G$-symmetric points becomes also a critical point of $J$ on the whole space $X$. This principle is generalized to the case where $J$ is not differentiable within the setting which does not require the full variational structure under the hypothesis that the action of $G$ is isometry or $G$ is compact.

Key words: symmetric criticality, subdifferential operator, group action, non-smooth functional, elliptic variational inequality

1 Introduction
Let $X$ be a Banach space on which $G$ linearly acts and let $J$ be a smooth $G$-invariant functional on $X$, that is, $J(gu) = J(u)$ for all $g \in G$ and $u \in X$. Let $\Sigma$ be the subspace consisting of all symmetric points with respect to $G$, i.e.,

*Partially supported by Waseda University Grant for Special Research Projects, #2002A-089.
†Partially supported by the Grant-in-Aid for Scientific Research, #12440051, the Ministry of Education, Culture, Sports and Technology, Japan and Waseda University Grant for Special Research Projects, #2001B-017.
\[ \Sigma = \{ u \in X : gu = u, \forall g \in G \}. \]

What we call the “Principle of Symmetric Criticality” (for smooth functionals) asserts that any critical points of \( J|_{\Sigma} \) (the restriction of \( J \) to \( \Sigma \)) give the critical points of \( J \) on the whole space \( X \). In the pioneering work of R. S. Palais [17], it is pointed out that an early implicit use of this principle can be found in [18] or [23] and an explicit reference to this principle can be found in Coleman’s paper [10], but whose sketch of a proof still contains some ambiguousness. Unfortunately this principle is not valid in general. In fact, Palais [17] gave some counterexamples where this principle does not hold. In spite of the presence of these pathological examples, however, he found out that this principle is valid in a reasonably broad context from the viewpoint of mathematical physics. In particular, he showed that this principle is valid if \( G \) is a compact Lie group or if \( X \) is a Hilbert space and \( G \) is isometric, (actually he discussed in a more general \( G \)-manifold setting). In his theory, however, one needs to work in the \( C^1 \)-category with full variational structure.

One of our main purposes of this paper is to discuss the case where the functional \( J \) to be considered need not be differentiable. More precisely, we consider the case where \( J \) is given as the sum of a \( G \)-invariant lower semicontinuous convex functional \( \varphi \) from \( X \) into \( (-\infty, +\infty] \) and a \( G \)-invariant \( C^1 \)-functional \( \Psi \) on \( X \). Then the “Principle” for this case should read

\[ \partial(\varphi|_{\Sigma})(u) + (\Psi|_{\Sigma})(u) \ni 0 \text{ in } \Sigma^* \implies \partial \varphi(u) + \Psi'(u) \ni 0 \text{ in } X^*. \quad (1) \]

Here \( \partial \varphi(u) \) denotes the subdifferential (a generalized Fréchet derivative) of \( \varphi \) at \( u \) and becomes a multivalued operator in general. Therefore, in order to investigate the validity of the “Principle” in this new version, we must work in the setting with multivaluedness nature, which makes the analysis much more difficult than in the classical setting. It seems that the study in this direction is not fully pursued yet. A related result can be found in Suzuki and Nagasaki [21].

Another main purpose of this paper is to present a more general form of the “Principle” whose setting does not require the full variational structure, in other word, the term \( (\Psi|_{\Sigma})(u) \) and \( \Psi'(u) \) in (1) could be replaced by more general operators which need not to be given as the derivatives of some functionals (see \( (P_1) \) in Section 3). This generalization enables us to apply the “Principle” not only for the elliptic equations with full variational structure but also for many partial differential equations without full variational structure, in particular, for some evolution equations, which will be discussed in our forthcoming paper [15].

This paper is composed of four sections. We shall first recall the “Principle” within the category of \( C^1 \)-functionals on Banach spaces in §2. Especially in §2.1, we give a new result in the classical setting, i.e., Theorem 2.2. We next discuss in §3 the new version of the “Principle” in the generalized setting mentioned above. In §3.1, we deals with the case where \( X \) and its dual \( X^* \) are both reflexive and strictly convex and \( G \) is isometric. Here a generalized version of Theorem 2.2 is discussed. To cope with the difficulty caused by the multivaluedness, we construct a suitable projection operator by using arguments based on the convex analysis and the geometry of Banach spaces. In §3.2, the case where \( G \) is
a compact topological group is treated. When $G$ is compact, the classical theory relies essentially on the averaging operator $A$ over $G$ constructed by using the normalized Harr measure on $G$. In our case, however, the adjoint operator $A^*$ of $A$ will play an important role such as the projection operator in the previous subsection. In the last section, we exemplify the applicability of our abstract setting to nonlinear elliptic equations by a variational inequality associated with the $p$-Laplacian in unbounded domains.

2 Principle for Smooth Functionals

Most of results in this section except in §2.1 is essentially contained in Palais [17]. However, since we are here concerned with the special (i.e., Banach space) setting, the arguments could be more direct and clear than those in [17], so we recall them in our setting for the sake of convenience.

Let $X$ be a real Banach space and let $X^*$ be its dual. The norms of $X$ and $X^*$ will be denoted by $\| \cdot \|$ and $\| \cdot \|_*$, respectively. We shall denote by $\langle \cdot, \cdot \rangle_X$ the duality pairing between $X$ and $X^*$, which will be simply denoted by $\langle \cdot, \cdot \rangle$ if no confusion arises.

Let $G$ be a group and let $\pi$ be a representation of $G$ over $X$, that is, $\pi(g)$ is a bounded linear operator in $X$ for each $g \in G$ and $\pi(e)u = u \quad \forall u \in X$,
$$\pi(g_1g_2)u = \pi(g_1)(\pi(g_2)u) \quad \forall g_1, g_2 \in G, \quad \forall u \in X,$$
where $e$ is the identity element of $G$. The representation $\pi_\ast$ of $G$ over $X^*$ is naturally induced by $\pi$ through the relation:
$$\langle \pi_\ast(g)v^\ast, u \rangle = \langle v^\ast, \pi(g^{-1})u \rangle \quad g \in G, \quad v^\ast \in X^*, \quad u \in X.$$
For simplicity, we shall often write $gu$ or $gu^\ast$ instead of $\pi(g)u$ or $\pi_\ast(g)v^\ast$ respectively. A function $h$ on $X$ (or $X^*$) is called $G$-invariant if $h(gu) = h(u) \quad \forall u \in X \quad (or \; \forall h(gu^\ast) = h(u^\ast) \quad \forall u^\ast \in X^*), \quad \forall g \in G$
and a subset $M$ of $X$ (or $M^*$ of $X^*$) is called $G$-invariant if $gM = \{gu : u \in M\} \subset M \quad (or \; gM^* \subset M^*) \quad \forall g \in G$.

The linear subspaces of $G$-symmetric points of $X$ and $X^*$ are defined as the common fixed points of $G$:
$$\Sigma = \{u \in X : gu = u \quad \forall g \in G\},$$
$$\Sigma_\ast = \{v^\ast \in X^* : gv^\ast = v^\ast \quad \forall g \in G\}.$$ 
Hence, by (2), we can easily see that $v^\ast \in X^*$ is symmetric if and only if it is a $G$-invariant functional. It is clear that $\Sigma$ and $\Sigma_\ast$ form closed linear subspaces of
X and X\(^{*}\) respectively, so \(\Sigma\) and \(\Sigma\)\(_{*}\) are regarded as Banach spaces with their induced topologies.

Let \(C^1_0(X)\) be the set of all \(G\)-invariant \(C^1\)-functional on \(X\). In this section, we consider the following principle:

(P\(_0\)) For all \(J \in C^1_0(X)\), it holds that \((J|\Sigma)'(u) = 0\) assures \(J'(u) = 0\).

Here \((J|\Sigma)'(u)\) and \(J'(u)\) denote the Fréchet derivatives of \(J|\Sigma\) and \(J\) at \(u\) in \(\Sigma\) and \(X\) respectively.

**Proposition 2.1** ([17, Proposition 4.2]) The principle (P\(_0\)) is valid if and only if

\[\Sigma_* \cap \Sigma^\perp = \{0\},\]

where \(\Sigma^\perp = \{v^* \in X^* : (v^*, u) = 0\ \forall u \in \Sigma\}\).

**Proof.** (If part) Suppose \(\Sigma_* \cap \Sigma^\perp = \{0\}\) and let \(u_0\) be a critical point of \(J|\Sigma\). We must show \(J'(u_0) = 0\). Since \(J(u_0) = J|\Sigma(u_0)\) and \(J(u_0 + v) = J|\Sigma(u_0 + v)\) for all \(v \in \Sigma\), we get

\[X \cdot (J'(u_0), v)_X = \Sigma \cdot (J|\Sigma)'(u_0), v)_\Sigma = 0\ \forall v \in \Sigma,

where \(\Sigma \cdot (\cdot, \cdot)_\Sigma\) denotes the duality pairing between \(\Sigma\) and its dual \(\Sigma^*\). This implies \(J'(u_0) \in \Sigma^\perp\). On the other hand, it follows from the \(G\)-invariance of \(J\) that

\[
(J'(gu), v) = \lim_{t \to 0} \frac{J(gu + tv) - J(gu)}{t} = \lim_{t \to 0} \frac{J(u + tg^{-1}v) - J(u)}{t} = (J'(u), g^{-1}v) = (gJ'(u), v)
\]

for all \(g \in G\) and \(v \in X\). This means \(J'\) is \(G\)-equivariant, i.e.,

\[
J'(gu) = gJ'(u) \quad \forall g \in G, \forall u \in X.
\]

Especially, since \(u_0 \in \Sigma\), we obtain \(gJ'(u_0) = J'(u_0)\) for all \(g \in G\), that is, \(J'(u_0) \in \Sigma_*\). Thus we conclude \(J'(u_0) \in \Sigma_* \cap \Sigma^\perp = \{0\}\), i.e., \(J'(u_0) = 0\).

(Only if part) Suppose that there exists a non-zero element \(v^* \in \Sigma_* \cap \Sigma^\perp\), and define \(J_*(\cdot)\) by \(J_*(u) = (v^*, u)\). Then it is clear that \(J_* \in C^1_0(X)\) and \((J_*')' = v^* \neq 0\), so \(J_*\) has no critical point in \(X\). On the other hand, the assumption \(v^* \in \Sigma^\perp\) implies \(v^*|_{\Sigma} = 0\), whence follows \((J_*|_{\Sigma})'(u) = 0\) for all \(u \in \Sigma\). This violates the principle (P\(_0\)). Therefore the condition \(\Sigma_* \cap \Sigma^\perp = \{0\}\) is necessary for the principle (P\(_0\)).

\[\blacksquare\]
2.1 The Isometry Case

In this subsection, we impose the following two assumptions:

(A.1) \( X \) is reflexive and strictly convex;

(A.2) The action of \( G \) over \( X \) is isometric, i.e.,
\[
\|gu\| = \|u\| \quad \forall g \in G, \forall u \in X.
\]

Then the following result holds.

**Theorem 2.2** Let (A.1) and (A.2) be satisfied. Then the principle \((P_0)\) is valid.

For the later use, we prepare a couple of propositions.

**Proposition 2.3** Assume that (A.2) is satisfied. Then the action of \( G \) over \( X \) becomes also isometric.

**Proof.** For all \( g \in G \) and \( v \in X \), we have
\[
\|gv\| = \|v\| = \|v\|.
\] Moreover, the above relation with \( g \) replaced by \( g^{-1} \) gives
\[
\|v\| = \|g^{-1}(gv)\| \leq \|gv\|.
\]

**Proposition 2.4** Assume that (A.2) is satisfied. Let \( F \) be the duality map from \( X \) into \( X^* \). Then for every \( v^* \in \Sigma \cap R(F) \), \( F^{-1}(v^*) \) forms a \( G \)-invariant set, i.e., \( gF^{-1}(v^*) \subset F^{-1}(v^*) \) for all \( g \in G \). Furthermore, if (A.1) is satisfied, then \( F^{-1}(\Sigma) \subset \Sigma \) holds.

**Proof.** Let \( v^* \in \Sigma \cap R(F) \) and take any \( v \in F^{-1}(v^*) \). Then, by (A.2), we get
\[
\|gv\| = \|v\| = \|v^*\|,
\]
\[
\langle v^*, gv \rangle = \langle g^{-1}v^*, v \rangle = \langle v^*, v \rangle = \|v^*\|^2,
\]
which implies \( gv \in F^{-1}(v^*) \), whence follows \( gF^{-1}(v^*) \subset F^{-1}(v^*) \) for all \( g \in G \).

We here recall that the reflexivity of \( X \) assures that \( F \) is surjective and the strict convexity of \( X \) assures that \( F \) is injective (see [9, Corollary II. 1.9]). That is, (A.1) assures that \( F^{-1} \) becomes a single-valued mapping defined on \( X^* \). Hence, the above argument says that \( gF^{-1}(v^*) = F^{-1}(v^*) \) for all \( g \in G \), i.e., \( F^{-1}(v^*) \in \Sigma \) for all \( v^* \in \Sigma \).

**Remark 1** For the case where \( X \) is not strictly convex, \( F^{-1}(v^*) \) could be \( G \)-invariant for every \( v^* \in \Sigma \). However, each element \( v \in F^{-1}(v^*) \) may not belong to \( \Sigma \), as is shown by the following counter example.
Counter example 1: Let $X = \mathbb{R}^3 = \{ u = (u_1, u_2, u_3) : u_i \in \mathbb{R}, \ i = 1, \ 2, \ 3 \}$ with the norm $|u|_X = |u|_\infty = \max_{1 \leq i \leq 3} |u_i|$. Then the dual norm of $X^* = \mathbb{R}^3$ becomes $|u^*|_{X^*} = |u|_1 = \sum_{i=1}^{3} |u_i|$. Let $\ell = \{ u_\ell = (0,0,t) : t \in \mathbb{R} \}$ and denote by $g_\theta$ the axial rotation in $\mathbb{R}^3$ with angle $\theta$, i.e., $g_\theta u = (u_1 \cos \theta - u_2 \sin \theta, u_1 \sin \theta + u_2 \cos \theta, u_3)$. Let $G_1$ be the group generated by $g_{\pi/2}$: $G_1 = \{ e, g_{\pi/2}, g_{\pi}, g_{3\pi/2} \}$. Then the action of $G_1$ becomes isometric. Furthermore it is clear that the action of $G_1$ on $X^*$ becomes the same as the action of $G_1$ on $X$ and that $\Sigma = \Sigma_* \subseteq \Sigma$. Let $v^* = (0,0,1) \in \Sigma_*$. Then it is easy to see that $F^{-1}(v^*) = \{(v_1, v_2, 1) : |v_1| \leq 1, \ |v_2| \leq 1 \}$. Hence $F^{-1}(v^*)$ is obviously $G_1$-invariant but each element $v = (v_1, v_2, 1)$ of $F^{-1}(v^*)$ is not $G_1$-symmetric except the case $v_1 = v_2 = 0$.

Now we are ready to prove Theorem 2.2.

**Proof of Theorem 2.2.** By virtue of Proposition 2.1, we only need to check the condition $\Sigma_* \cap \Sigma^\perp = \{ 0 \}$.

Let $v^* \in \Sigma_* \cap \Sigma^\perp$. Then Proposition 2.4 with the fact $v^* \in \Sigma_*$ assures $F^{-1}(v^*) \in \Sigma$. Therefore, by using the fact that $v^* \in \Sigma^\perp$, we obtain $\|v^*\|_2^2 = \langle v^*, F^{-1}(v^*) \rangle = 0$, i.e., $v^* = 0$.

\[ \langle v^*, Au \rangle = \int_G \langle v^*, gu \rangle d\mu(g) \ orall v^* \in X^*, \quad (4) \]

where $\mu$ is the normalized Harr measure on $G$. The mapping $A$ is called the averaging over $G$ and has the following properties:

- $A$ is a continuous linear projection from $X$ onto $\Sigma$.
- If $K$ is a $G$-invariant closed convex subset of $X$, then $A(K) \subseteq K$.

As for these properties, we refer the reader to the book of Vanderbauwhede [22, Section 2.5] (see also [19, Chapter 3 and 5]).

**Theorem 2.5 (\cite[Theorem 5.1]{17})** If (A.3) is satisfied, then $(P_0)$ is valid.

**Proof.** We check the condition $\Sigma_* \cap \Sigma^\perp = \{ 0 \}$ again. Let $v^* \in \Sigma_* \cap \Sigma^\perp$ and suppose $v^* \neq 0$. Since $v^* \in \Sigma_*$, the hyperplane $H = \{ u : \langle v^*, u \rangle = 1 \}$ becomes a non-empty $G$-invariant closed convex subset of $X$. Then, for any $u \in H$, we have $Au \in H \cap \Sigma$ and hence $\langle v^*, Au \rangle = 0$ since $v^* \in \Sigma^\perp$. This contradicts the fact that $Au \in H$.

\[ \]
3 Principle for Subdifferentials

Let \( \Phi(X) \) be the set of all proper lower semicontinuous convex functions \( \varphi \) from \( X \) into \((-\infty, +\infty]\), where “proper” means the effective domain \( D(\varphi) = \{ u \in X : \varphi(u) < +\infty \} \) of \( \varphi \) is not empty. For \( u \in D(\varphi) \), the subdifferential \( \partial \varphi(u) \) of \( \varphi \) at \( u \) is defined by

\[
\partial \varphi(u) = \{ u^* \in X^* : \varphi(v) - \varphi(u) \geq \langle u^*, v - u \rangle \quad \forall v \in X \}.
\]

Then, as is well known, \( \partial \varphi \) is a maximal monotone operator from \( X \) into \( X^* \) (see Barbu [4] and Brézis [7]).

Let \( \Phi_G(X) \) be the set of all \( G \)-invariant functionals belonging to \( \Phi(X) \), and let \( \Gamma_G(X^*) \) be the set of all \( G \)-invariant weakly* closed convex subset of \( X^* \).

In this section, as a generalization of the classical “principle of symmetric criticality”, we introduce the following principle \((P)\).

\((P_1)\) For all \( \varphi \in \Phi_G(X) \) and all \( K \in \Gamma_G(X^*) \), it holds that

\[
\partial(\varphi|_\Sigma)(u) \cap K|_\Sigma \neq \emptyset \implies \partial \varphi(u) \cap K \neq \emptyset,
\]

where \( K|_\Sigma = \{ u^*|_\Sigma : u^* \in K \} \) with \( \Sigma \cdot \langle u^*|_\Sigma, u \rangle|_\Sigma := \langle u^*, u \rangle|_X (u \in \Sigma) \).

Apparantly, this principle \((P_1)\) seems to have no recognizable relation to the classical principle \((P_0)\). However it turns out that \((P_1)\) gives a generalization of \((P_0)\). To see this, we first note the following fact:

\[
(J|_\Sigma)'(u) = (J'(u))|_\Sigma \quad \forall J \in C^1(X), \forall u \in \Sigma.
\]  

(5)

Indeed, \( J'(u) \) satisfies

\[
J(u + h) = J(u) + \langle J'(u), h \rangle + o(\|h\|) \quad \forall h \in X,
\]

then \((J'(u))|_\Sigma\) satisfies

\[
J(u + h) = J(u) + \Sigma \cdot \langle (J'(u))|_\Sigma, h \rangle|_\Sigma + o(\|h\|) \quad \forall h \in \Sigma.
\]

Noticing that \( u, u + h \in \Sigma \) impy \( J(u + h) = J|_\Sigma(u + h) \) and \( J(u) = J|_\Sigma(u) \), we get \((J|_\Sigma)'(u) = (J'(u))|_\Sigma\).

Here, let \( J \in C^1_G(X) \) and put \( K = \{-J'(u)\} \) with \( u \in \Sigma \), then by virtue of (3), we get \( K \in \Gamma_G(X^*) \). Therefore, in view of (5), we find that \((P_1)\) yields

\((P_1)'\) For all \( \varphi \in \Phi_G(X) \) and all \( J \in C^1_G(X) \), it holds that

\[
\partial(\varphi|_\Sigma)(u) + (J|_\Sigma)'(u) \ni 0 \implies \partial \varphi(u) + J'(u) \ni 0.
\]

Here, in particular, take \( \varphi \equiv 0 \), then \( \partial(\varphi|_\Sigma)(u) = \partial \varphi(u) = \{0\} \). Hence \((P_1)\) reads \( (J|_\Sigma)'(u) = 0 \implies J'(u) = 0^0 \). Thus \((P_1)'\) with \( \varphi \equiv 0 \) gives the classical principle of symmetric criticality \((P_0)\).

Furthermore, if we take \( K = -J'(u) + \partial \psi(u) \) with \( J \in C^1_G(X), \psi \in \Phi_G(X) \), and \( u \in \Sigma \), then (3) and Proposition 3.1 below assure that \( K \in \Gamma_G(X^*) \). Therefore, this time, \((P_1)\) yields
(P1)^\prime \prime \prime \prime For all \( \varphi, \psi \in \Phi_G(X) \) and all \( J \in C^1_G(X) \), it holds that  
\[ \partial(\varphi|_\Sigma)(u) + (J|_\Sigma)'(u) - \partial(\psi|_\Sigma)(u) \geq 0 \implies \partial\varphi(u) + J'(u) - \partial\psi(u) \geq 0, \]
provided that \( \partial(\psi|_\Sigma)(u) = (\partial\psi(u))|_\Sigma \) (see Lemma 3.7 and Remark 4).

If one is faithful to the usage of the word “criticality”, it is plausible to consider only the case where the set \( K \) has also a variational structure such that \( K = T(u) \) is given as a (generalized) derivative of a functional of \( u \) as above. However, our abstract setting allows us to deal with the case which \( K \) does not have any variational structures. Indeed (P1) can be applied to some parabolic type equations (lacking the full variational structure), which will be discussed in our forth coming paper [?].

One of the main difficulties in discussing the symmetric criticality for the equations involving the subdifferentials lies in the multivaluedness of the subdifferentials. Nevertheless, it shows a close analogy with the classical case. We begin with the analogue of (3), the \( G \)-equivariant property.

**Proposition 3.1** For all \( \varphi \in \Phi_G(X) \), the subdifferential \( \partial\varphi \) of \( \varphi \) is \( G \)-equivariant, i.e.,  
\[ \partial\varphi(gu) = g\partial\varphi(u) \quad \forall g \in G, \quad \forall u \in X. \]

**Proof.** We first prove \( \partial\varphi(gu) \subset g\partial\varphi(u) \). Let \( v^* \in \partial\varphi(gu) \). Then we have  
\[ \varphi(v) - \varphi(u) = \varphi(gu) - \varphi(u) \geq \langle v^*, gv - gu \rangle = \langle g^{-1}v^*, v - u \rangle \]
for all \( v \in X \). This implies \( g^{-1}v^* \in \partial\varphi(u) \) and hence \( v^* \in g\partial\varphi(u) \).

Moreover, the above relation with \( g \) replaced by \( g^{-1} \) gives  
\[ g\partial\varphi(u) = g\partial\varphi(g^{-1}gu) \subset gg^{-1}\partial\varphi(gu) = \partial\varphi(gu), \]
which completes the proof. 

**Remark 2** It follows from the proposition above that if \( u \in \Sigma \), then \( \partial\varphi(u) \in \Gamma_G(X^*) \). Especially if \( \partial\varphi \) is single-valued, then \( \partial\varphi(u) \in \Sigma_* \). However, in general, each element of \( \partial\varphi(u) \) need not be \( G \)-symmetric (cf. (3)) as is shown in the following counter example.

**Counter example 2.** Let \( X = \mathbb{R}^3 \) with the norm \( \|u\| = \|u\|_2 = (\sum_{i=1}^3 |u_i|^2)^{1/2} \) and take \( \ell \) and \( g_0 \) as in Counter example 1. Let \( G_2 = \{g_0 : \theta \in [0, 2\pi)\} \). Then \( G_2 \) becomes a compact and isometric group acting on the Hilbert space \( \mathbb{R}^3 \). Furthermore it is clear that \( \Sigma = \Sigma_* = \ell, \Sigma^\perp = \{u^\perp = (u_1, u_2, 0) : u_1, u_2 \in \mathbb{R}\} \), and \( \Sigma_* \cap \Sigma^\perp = \{0\} \). Let \( \varphi \) be the indicator function of \( \Sigma = \ell, \) i.e., \( \varphi(u) = 0 \) if \( u \in \Sigma \) and \( \varphi(u) = +\infty \) if \( u \in \mathbb{R}^3 \setminus \Sigma \). Then \( D(\varphi) = \Sigma \) and \( \partial\varphi(u) = \Sigma^\perp \) for all \( u \in \Sigma \). Hence \( \partial\varphi(u) \) is obviously invariant under the action of \( G_2 \), but each element \( u^\perp = (u_1, u_2, 0) \) of \( \partial\varphi(u) \) is not \( G_2 \)-symmetric except the case \( u^\perp = 0 \).
This counter example shows that even if assumptions (A.1)’ (introduced in §3.1), (A.2), and (A.3) are satisfied, each element of $\partial \varphi(u)$ with $u \in \Sigma$ may not be $G$-symmetric. Because of this fact, the previous argument for regular (single-valued) operators does not work for the multivalued operators any more. To cope with this difficulty, we here introduce a projection $P$ from $X^*$ onto $\Sigma_*$ satisfying the following property:

(a) For all $K \in \Gamma_G(X^*)$, it holds that $P(K) \subset K$.

**Proposition 3.2** Suppose that $\Sigma_* \cap \Sigma^\perp = \{0\}$ holds and that there exists a linear projection $P$ from $X^*$ onto $\Sigma_*$ satisfying (a). Moreover, assume that $\partial(\varphi + I_\Sigma) = \partial \varphi + \partial I_\Sigma$ holds, where $I_\Sigma$ is the indicator function of $\Sigma$, i.e., $I_\Sigma(u) = 0$ if $u \in \Sigma$ and $I_\Sigma(u) = +\infty$ if $u \in X \setminus \Sigma$. Then the principle $(P_1)$ is valid.

**Proof.** Let $\varphi \in \Phi_G(X)$ and $K \in \Gamma_G(X^*)$, and assume that

$$\partial(\varphi|_\Sigma)(u) \cap K|_\Sigma \neq \emptyset.$$ (6)

We first prove

$$\left(\partial \varphi(u) + \partial I_\Sigma(u)\right) \cap K \neq \emptyset.$$ (7)

Indeed, by (6), there exist $v^* \in K$ such that $v^*|_\Sigma \in \partial(\varphi|_\Sigma)(u)$. This implies

$$u \in D(\varphi|_\Sigma) \text{ and } \varphi|_\Sigma(w) - \varphi|_\Sigma(u) \geq \varphi|_\Sigma(v^*|_\Sigma; w - u) \quad \forall w \in \Sigma,$$

or

$$u \in \Sigma \cap D(\varphi) \text{ and } \varphi(w) - \varphi(u) \geq \langle v^*, w - u \rangle_X \quad \forall w \in \Sigma.$$

Therefore we have $v^* \in \partial(\varphi + I_\Sigma)(u) = \partial \varphi(u) + \partial I_\Sigma(u)$. Thus (7) is verified.

By (a), we have $Pw^* + Pz^* \in K$. Since $u \in \Sigma$, by Proposition 3.1, we get $\partial \varphi(u) \in \Gamma_G(X^*)$ and hence, by (a), $Pw^* \in \partial \varphi(u)$. Thus in order to conclude $\partial \varphi(u) \cap K \neq \emptyset$, it suffices to show $Pz^* = 0$.

Since $P$ is a projection onto $\Sigma_*$, we have $Pz^* \in \Sigma_*$. On the other hand $z^* \in \Sigma^\perp$ follows from the fact that the range of $\partial I_\Sigma$ coincides with $\Sigma^\perp$. Hence, by using (a) again, we have $Pz^* \in \Sigma^\perp$ since it is easily seen that $\Sigma^\perp$ is a $G$-invariant weakly* closed linear subspace of $X^*$. Consequently, we deduce $Pz^* = 0$ from the assumption $\Sigma_* \cap \Sigma^\perp = \{0\}$. $\blacksquare$.

### 3.1 The Isometry Case

We here assume (A.2) and the following (A.1)', a little bit stronger than (A.1).

(A.1)' $X$ is reflexive and the norms of $X$ and $X^*$ are both strictly convex.

Then our main result here is stated as follows.

**Theorem 3.3** Assume that (A.1)' and (A.2) are satisfied. Then the principle $(P_1)$ is valid.
Proof. We first construct the projection \( P : X^* \to \Sigma_* \) which satisfies \((\alpha)\) and next show that \( \partial(\varphi + I_{\Sigma^*}) = \partial\varphi + \partial I_{\Sigma^*} \), that is, \( \partial\varphi + \partial I_{\Sigma^*} \) is maximal monotone. To carry out this program, we prepare several results.

**Proposition 3.4** Let \((A.1)'\) and \((A.2)\) be satisfied, and let \( F \) be the duality map form \( X \) into \( X^* \). Then it holds that \( F(\Sigma) = \Sigma_* \).

**Proof of Proposition 3.4.** Under \((A.1)'\), the duality map \( F \) becomes a bijection from \( X \) to \( X^* \) and the same argument as that in the proof of Proposition 2.4 gives
\[
F(gu) = gF(u) \quad \forall g \in G, \quad \forall u \in X
\]
and hence \( F(\Sigma) \subset \Sigma_* \). Furthermore \( F^{-1}(\Sigma_*) \subset \Sigma \) is assured by Proposition 2.4. Thus we obtain \( F(\Sigma) = \Sigma_* \). \( \blacksquare \)

**Proposition 3.5** Assume that \((A.1)'\) and \((A.2)\) are satisfied. Then we have
\[
X^* = \Sigma_* \oplus \Sigma^\perp,
\]
i.e., \( \Sigma_* \) and \( \Sigma^\perp \) are topological complements to each other.

**Remark 3** By substituting \( X \) for \( X^* \), we also have
\[
X = \Sigma \oplus (\Sigma_*)\perp,
\]
where \( (\Sigma_*)\perp = \{ u \in X : (v^*, u) = 0 \ \forall v^* \in \Sigma_* \} \).

**Proof of Proposition 3.5.** Let \( v^*_0 \in X^* \). Since \( \Sigma_* \) and \( \Sigma^\perp \) are closed and \( \Sigma_* \cap \Sigma^\perp = \{0\} \) (by Theorem 2.2), we have only to show that \( v^*_0 \) can be decomposed into the sum of two elements in \( \Sigma_* \) and \( \Sigma^\perp \).

To do this, let us introduce the functional \( \rho \) on \( X^* \) defined by
\[
\rho(z^*) := \frac{1}{2}||z^* - v^*_0||^2_{\Sigma^*}, \quad z^* \in X^*.
\]

Since \( X^* \) is reflexive and strictly convex, there exists a unique minimizer \( z^*_0 \) of \( \rho|_{\Sigma^\perp} \), that is, the nearest point of \( v^*_0 \) in \( \Sigma^\perp \). We here recall the fact that the norm of \( X^* \) is Gâteaux differentiable except at the origin since \( X \) is strictly convex (see Köthe [16, §26]). Hence \( \rho \) becomes everywhere Gâteaux differentiable. Moreover its Gâteaux derivative \( d\rho(z^*_0) \) coincides with \( F^{-1}(z^*_0 - v^*_0) \) (cf. Barbu [4, II.2.2., Example 2]). So, for any \( z^* \in \Sigma^\perp \) and \( t \in \mathbb{R} \), we get
\[
0 \leq \rho(z^*_0 + tz^*) - \rho(z^*_0) = \langle tz^*, F^{-1}(z^*_0 - v^*_0) \rangle + o_{z^*}(t),
\]
where \( o_{z^*}(t) \) denotes the remainder term depending on \( z^* \) such that
\[
\lim_{t \to 0} o_{z^*}(t)/t = 0.
\]
Then dividing both sides by \( t > 0 \), \( t < 0 \) and letting \( t \to +0 \), \( t \to -0 \), we deduce
\[
\langle z^*, F^{-1}(z^*_0 - v^*_0) \rangle = 0 \quad \forall z^* \in \Sigma^\perp,
\]

or \( F^{-1}(z_0^* - v_0^*) \in (\Sigma^\bot)_\perp = \{ u \in X : \langle v^*, u \rangle = 0 \forall v^* \in \Sigma^\bot \} \) (cf. [1, Corollary 1.5]). Since \((\Sigma^\bot)_\perp\) coincides with \(\Sigma\) ([6, Proposition II.12]), Proposition 3.4 says that \(z_0^* - v_0^* = F(F^{-1}(z_0^* - v_0^*)) \in \Sigma_s\). Hence \(v_0^*\) is decomposed as follows:
\[
v_0^* = (v_0^* - z_0^*) + z_0^*, \quad v_0^* - z_0^* \in \Sigma_s, \quad z_0^* \in \Sigma^\bot,
\]
which completes the proof.

Thus we can define two linear projections \(P: X^* \to \Sigma_s\) and \(Q: X^* \to \Sigma^\bot\) by
\[
P: v_0^* \mapsto v_0^* - z_0^*; \quad Q: v_0^* \mapsto z_0^*.
\]

**Lemma 3.6** The projection \(P\) defined above satisfies \((\alpha)\).

**Proof of Lemma 3.6.** Let \(K \in \Gamma_G(X^*)\) and \(v_0^* \in K\). By Proposition 3.5, \(v_0^*\) is decomposed as follows:
\[
v_0^* = w_0^* + z_0^*, \quad w_0^* = P v_0^* \in \Sigma_s, \quad z_0^* = Q v_0^* \in \Sigma^\bot.
\]
We are going to show \(w_0^* \in K\).

Suppose otherwise, i.e., \(w_0^* \notin K\). Then, by (A.1)', there exists the nearest point \(v_1^*\) of \(w_0^*\) in \(K \cap (w_0^* + \Sigma^\bot)\) (Note that this set is not empty since \(v_0^* \in K \cap (w_0^* + \Sigma^\bot)\)). We write
\[
v_1^* = w_0^* + z_1^*, \quad z_1^* \in \Sigma^\bot.
\]
Since \(w_0^* \notin K\), we have \(z_1^* \neq 0\). Hence, by Proposition 3.5, \(z_1^* \notin \Sigma_s\). Therefore there exists a \(g \in G\) such that \(g z_1^* \neq z_1^*\). Put
\[
z_2^* = \frac{1}{2}(z_1^* + g z_1^*)
\]
for such \(g\). Then \(z_2^* \in \Sigma^\bot\) holds, since \(\Sigma^\bot\) is a \(G\)-invariant subspace. Moreover, we get \(\|z_2^*\|_s < \|z_1^*\|_s\), since \(\|g z_1^*\|_s = \|z_1^*\|_s\) and \(X^*\) is strictly convex. Put
\[
v_2^* = w_0^* + z_2^* \quad (\in w_0^* \cap \Sigma^\bot).
\]
Then it is easy to see
\[
v_2^* = \frac{1}{2} v_1^* + \frac{1}{2} g v_1^*.
\]
Thus, since \(K\) is \(G\)-invariant and convex, we find that \(v_2^* \in K\) and \(\|z_2^*\|_s = \|v_2^* - w_0^*\|_s < \|v_1^* - w_0^*\|_s = \|z_1^*\|_s\), which contradicts the definition of \(v_1^*\).

The following lemma completes the proof of Theorem 3.3.

**Lemma 3.7** Let the same assumptions in Theorem 3.3 be satisfied. Then \(\partial \varphi + \partial I_{\Sigma}\) becomes maximal monotone.

To prove this lemma, we need the following fact which is well known for the case where \(X\) is a Hilbert space ([7, Theorem 9]).
Lemma 3.8 Assume that (A.1)' is satisfied. Let $A : X \to X^*$ be a maximal monotone operator, $J_A^\lambda$ its resolvent, and $\psi \in \Phi(X)$. Suppose that there exists $C > 0$ such that

$$
\psi(J_A^\lambda u) \leq \psi(u) + C\lambda \quad \forall u \in D(\psi), \quad \forall \lambda > 0.
$$

Then $A + \partial \psi$ is maximal monotone.

Proof of Lemma 3.8. We first recall the following facts (see [4, Chap. II §1] or [8]):

- A monotone operator $B : X \to X^*$ is maximal if and only if $R(F + B) = X^*$.
- If $A$ is maximal monotone, then $A_{\lambda}$ becomes bounded, monotone, and demicontinuous from $X$ to $X^*$. Hence $A_{\lambda} + \partial \psi$ becomes maximal monotone,

where $A_{\lambda}u = \lambda^{-1}F(u - J_A^\lambda u)$ and $J_A^\lambda u$ satisfies

$$
F(J_A^\lambda u - u) + \lambda A(J_A^\lambda u) \ni 0.
$$

Thus for an arbitrary element $v^* \in X^*$, there exists a unique element $u_{\lambda} \in X$ satisfying

$$
F(u_{\lambda}) + A_{\lambda}u_{\lambda} + \partial \psi(u_{\lambda}) \ni v^* \quad \forall \lambda > 0.
$$

We are going to show below that $u_{\lambda}$ converges weakly to $u \in X$ satisfying

$$
F(u) + Au + \partial \psi(u) \ni v^*.
$$

Thus $R(F + A + \partial \psi) = X^*$, i.e., $A + \partial \psi$ is maximal monotone (cf. [8, Theorem 2.1]).

By (10) and the definition of $\partial \psi$, we have

$$
\psi(w) - \psi(u_{\lambda}) \geq \langle v^* - F(u_{\lambda}) - A_{\lambda}u_{\lambda}, w - u_{\lambda} \rangle \quad \forall w \in D(\psi).
$$

Putting $w = J_A^\lambda u_{\lambda}$, we get, by (9)

$$
C\lambda \geq \psi(J_A^\lambda u_{\lambda}) - \psi(u_{\lambda}) \geq \langle v^* - F(u_{\lambda}) - A_{\lambda}u_{\lambda}, -\lambda F^{-1}(A_{\lambda}u_{\lambda}) \rangle.
$$

Hence, from the fact that $\|F(z)\| = \|z\|$, we deduce

$$
\|A_{\lambda}u_{\lambda}\|^2 \leq C + \left(\|v^*\| + \|u_{\lambda}\|\right)\|A_{\lambda}u_{\lambda}\|.
$$

On the other hand, putting $w = w_0$ in (11) for some $w_0$ fixed in $D(A) \cap D(\psi)$, we have

$$
\psi(w_0) - \psi(u_{\lambda}) \geq \langle v^*, w_0 - u_{\lambda} \rangle - \langle F(u_{\lambda}), w_0 \rangle + \|u_{\lambda}\|^2
+ \langle A_{\lambda}w_0 - A_{\lambda}u_{\lambda}, w_0 - u_{\lambda} \rangle - \langle A_{\lambda}u_0, w_0 - u_{\lambda} \rangle.
$$
Therefore, noting the facts that $A_\lambda$ is monotone, $\psi(u_\lambda) \geq -C_1\|u_\lambda\| - C_2$, and $\|A_\lambda u_0\|$ is bounded, we can derive the boundedness of $\|u_\lambda\|$ independent of $\lambda$. Consequently, by (12), we find that $\|A_\lambda u_\lambda\|$ is also bounded.

We have thus proved that $A + \partial \psi$ is maximal monotone.

**Proof of Lemma 3.7.** We apply the fact above with $A = \partial \varphi$ and $\psi = I_\Sigma$. Let $\lambda > 0$ and let $J_\lambda$ be the resolvent of $\partial \varphi$. It then suffices to show $J_\lambda(\Sigma) \subset \Sigma$, which implies (9) with $C = 0$.

By the definition of $J_\lambda$, we have

$$F(J_\lambda u - u) + \lambda \partial \varphi(J_\lambda u) \ni 0 \quad \forall u \in X.$$ 

Multiplying this by $g \in G$, by (8) and Proposition 3.1, we obtain

$$F(g J_\lambda u - gu) + \lambda \partial \varphi(g J_\lambda u) \ni 0 \quad \forall g \in G, \forall u \in X.$$ 

Therefore, by the definition of $J_\lambda$, we get

$$J_\lambda(g u) = g J_\lambda u \quad \forall g \in G, \forall u \in X.$$ 

Especially if $u \in \Sigma$, then $J_\lambda u = g J_\lambda u$ for all $g \in G$, that is, $J_\lambda u \in \Sigma$. This completes the proof of the lemma (and hence Theorem 3.3).

**Remark 4** Let $(\partial \varphi)|_\Sigma = \{[u, f] \in \Sigma \times \Sigma^* | \exists h \in \partial \varphi(u) \text{ s.t. } h|_\Sigma = f\}$. Then it is easy to check $(\partial \varphi)|_\Sigma \subset \partial(\varphi|_\Sigma)$. By Asakawa [2, Corollary 2.2], $\partial \varphi + \partial I_\Sigma$ is maximal monotone if and only if $(\partial \varphi)|_\Sigma = \partial(\varphi|_\Sigma)$, i.e., $(\partial \varphi)|_\Sigma$ is maximal monotone in $\Sigma \times \Sigma^*$.

### 3.2 The Compact Case

In this subsection, we always assume (A.3).

To prove $\Sigma_* \cap \Sigma^\perp = \{0\}$, in the classical setting, the averaging operator $A$ over $G$ played an important role in §2.2. On the other hand, the crucial point for verifying the principle $(P_1)$ in our setting is the construction of the operator $P$ which maps $X^*$ into $\Sigma_*$ and satisfies the property $(a)$. As for the compact case in our new setting, the adjoint operator $A^*$ of $A$ plays the same important role as $P$ in the previous subsection.

**Lemma 3.9** The adjoint operator $A^*$ of $A$ is a mapping from $X^*$ into $\Sigma_*$. If $K \in \Gamma_G(X^*)$, then $A^*(K) \subset K$.

**Proof.** We first prove that $A^* v^* \in \Sigma_*$ for all $v^* \in X^*$. By the right invariance of the Harr measure [19, Theorem 5.14] and (4), we easily get

$$Agu = Au \quad \forall g \in G, \forall u \in X.$$ 

Therefore

$$\langle g A^* v^*, u \rangle = \langle v^*, Ag^{-1} u \rangle = \langle v^*, Au \rangle = \langle A^* v^*, u \rangle$$
for all \( g \in G \) and \( u \in X \), that is, \( A^*v^* \in \Sigma_* \).

We next prove \( A^*(K) \subset K \). Suppose that there exists an element \( v^* \in K \) such that \( A^*v^* \notin K \). We apply the Hahn-Banach theorem [14, Corollary 14.4] in \( X^* \) with weak* topology \( \sigma(X^*, X) \). Then there exist \( u \in X \), \( c \in \mathbb{R} \), and \( \varepsilon > 0 \) such that

\[
\langle A^*v^*, u \rangle \leq c - \varepsilon < c \leq \langle w^*, u \rangle \quad \forall w^* \in K.
\]

By putting \( w^* = g^{-1}v^* \in K \) for all \( g \in G \), we get

\[
\langle v^*, Au \rangle \leq c - \varepsilon < c \leq \langle v^*, gu \rangle \quad \forall g \in G,
\]

which contradicts (4).

Thus, with the aid of Theorem 2.5 and Proposition 3.2, we obtain the following result.

**Theorem 3.10** Assume that (A.3) is satisfied and \( \partial \varphi + \partial I_\Sigma \) is maximal monotone. Then the principle \( (P_1) \) is valid.

The maximal monotonicity of \( \partial \varphi + \partial I_\Sigma \) is assumed as an important hypothesis in the theorem above. However the verification of it is not so easy a task, except under some special setting such as in Lemma 3.7. Here we note that the compact case can be reduced to the isometry case in Theorem 2.2, provided that \( X \) is reflexive by using the renorming technique, which is suggested to us by H. Asakawa.

**Proposition 3.11** Assume that \( X \) is reflexive and (A.3) is satisfied. Then there exists an equivalent norm of \( X \) with which (A.1)' and (A.2) are satisfied.

**Proof.** By the result of Asplund [3], we can choose an equivalent norm of \( X \) with which \( X \) and \( X^* \) are both strictly convex. We first claim that there exists a constant \( C > 0 \) such that

\[
\frac{1}{C} \|u\| \leq \|gu\| \leq C \|u\| \quad \forall g \in G, \; u \in X. \tag{13}
\]

Indeed, since \( G \) is a compact group, we have

\[
\sup_{g \in G} \|gu\| < \infty \quad \forall u \in X.
\]

So it follows from the Banach-Steinhaus theorem that there exist a constant \( C > 0 \) such that

\[
\|gu\| \leq C \|u\| \quad \forall g \in G, \; u \in X.
\]

Noticing that \( u = g^{-1}gu \), we obtain (13).

Let \( Y := L^2(G; X) \) with the norm

\[
\|y\|_Y = \left( \int_G \|y(g)\|^2 d\mu(g) \right)^{1/2},
\]

14
For $u \in X$, we define $\widehat{u} \in Y$ by $\widehat{u}(g) = gu$ ($g \in G$) and put $\widehat{X} = \{ \widehat{u} : u \in X \}$.
Then, by (13), $\widehat{X}$ is isomorphic to $X$ and the norm
\[
|||u||| := |||\widehat{u}|||_Y = \left( \int_G ||gu||^2 d\mu(g) \right)^{1/2} \quad (u \in X)
\]
is equivalent to that of $X$.

We are going to show that $(X, ||| \cdot |||)$ satisfies (A.1)$'$ and (A.2). We here recall the following facts:

- A Banach space $E$ is strictly convex if and only if
  \[
  \frac{||x + y||^2}{2} \leq \frac{1}{2}(||x||^2 + ||y||^2) \quad \forall x, y \in E, \ x \neq y
  \]
  ([5, Part 3, Chap. I, Proposition 1]).

- If $E$ satisfies (A.1)$'$, then any closed linear subspace of $E$ also satisfies (A.1)$'$ ([16, Chap. 5, §26, 3(3), (4)]).

Using the former fact, we can easily show that $Y$ and $Y^* = L^2(G; X^*)$ are strictly convex. Therefore, by the latter fact, $(X, ||| \cdot |||)$ ($\simeq \widehat{X}$) satisfies (A.1)$'$.

Finally, we can see that the action of $G$ is isometric with respect to the norm $||| \cdot |||$ by the right invariance of the Harr measure.

Thus, if $X$ is reflexive and (A.3) is satisfied, then by combining Proposition 3.11 and Lemma 3.7, we can conclude that $\partial \varphi + \partial I_\Sigma$ becomes maximal monotone. Hence, from Theorem 3.10, we deduce the following result.

**Theorem 3.12** Assume that $X$ is reflexive and (A.3) is satisfied. Then the principle $(P_1)$ is valid.

## 4 Application

In Mechanics and Physics, one often encounters some problems which can be expressed in terms of inequalities in situations where the constraints, the equations of state, the physical lows change when certain thresholds are crossed or attained. And across or on these thresholds, there should normally appear some non-differentiable structure in the system. Nevertheless, many of them can be formulated as "variational inequalities", which can be well analyzed in a weak form (see Duvant-Lions [12]). Here as a typical example of constraint problems, we consider the following variational inequality:

Find $u \in K$ such that
\[
\int_{\Omega} |\nabla u|^{p-2} \nabla u \cdot \nabla (v - u) + \lambda \int_{\Omega} u^{p-1}(v - u) \geq \int_{\Omega} u^{q-1}(v - u) + \int_{\Omega} f(v - u), \quad \forall v \in K,
\]
(14)
where \( \lambda \in \mathbb{R} \), \( 1 < p < q \), \( f \in L^{p/(p-1)}(\Omega) \) (\( \Omega \) is a domain in \( \mathbb{R}^N \)), and
\[
K = \{ v \in W_0^{1,p}(\Omega) : v \geq 0 \text{ a.e. in } \Omega \}.
\]

Roughly speaking, this problem gives a weak form to find a solution \( u \) satisfying:
\[
- \text{div}(\nabla u|^{p-2}\nabla u) + \lambda u^{p-1} = u^{q-1} + f
\]
under the constraint \( u \in K \). Here the operator \( \Delta_p : u \mapsto \text{div}(\nabla u|^{p-2}\nabla u) \) coincides with the usual Laplacian when \( p = 2 \), and is often discussed as a prototype model in the theory of non-Newtonian fluids (see Kalashinikov [13]) or climatology (see Daz [11]).

Let \( \Psi : W_0^{1,p}(\Omega) \to \mathbb{R} \) be given by
\[
\Psi(u) = \frac{1}{p} \int_{\Omega} (|\nabla u|^p + \lambda |u|^p) - \frac{1}{q} \int_{\Omega} |u|^q - \int_{\Omega} fu
\]
and let \( I_K \) be the indicator function of \( K \). Then \( u \in W_0^{1,p}(\Omega) \) is a solution of (14) if and only if \( u \) is a critical point of \( \Psi + I_K \), that is, \( \Psi'(u) + \partial I_K(u) \ni 0 \).

When \( \Omega \) is bounded and \( p = 2 \), the existence of solution for this problem is shown in [20]. For the case where \( \Omega \) is unbounded, however, (even if \( p = 2 \)) the existence of solution for this problem is highly nontrivial because of the lack of compactness of the embedding: \( W_0^{1,p}(\Omega) \subset L^q(\Omega) \). On the other hand, it is known that the compactness of the above embedding recovers for suitable subspaces consisting of functions which possess high symmetry (see Proposition 4.2 below). So we may expect the possibility of finding a solution for this problem in such symmetric subspaces, where our abstract results are applicable.

Let \( \Omega \) be an (unbounded) domain in \( \mathbb{R}^N \) with smooth boundary \( \partial \Omega \) and let \( G \) be a subgroup of \( O(N) \) whose elements leave \( \Omega \) invariant: \( g(\Omega) = \Omega \) for all \( g \in G \). We assume that \( \Omega \) is compatible with \( G \) (see [24]), that is, for some \( r > 0 \)
\[
m(y, r, G) \to \infty \quad \text{as} \quad \text{dist}(y, \Omega) \leq r, \quad |y| \to \infty \quad (15)
\]
where
\[
m(y, r, G) = \sup \left\{ n \in \mathbb{N} : \exists g_1, g_2, \ldots, g_n \in G \text{ s.t. } B(g_jy, r) \cap B(g_ky, r) = \emptyset \text{ if } j \neq k \right\}.
\]

Let \( X = W_0^{1,p}(\Omega) \) and define a representation of \( G \) over \( X \) as follows:
\[
(\pi(g)u)(x) = u(g^{-1}x), \quad g \in G, \ u \in X, \ x \in \Omega.
\]
As usual we shall write \( gu \) in place of \( \pi(g)u \). A function \( u \) defined on \( \Omega \) is said to be \( G \)-invariant if
\[
u(gx) = u(x), \quad \forall g \in G, \ a.e. \ x \in \Omega.
\]
Then \( u \in X \) is \( G \)-invariant if and only if
\[
uu \in W_0^{1,p}(\Omega) := \{ u \in X | \ u = gu, \ \forall g \in G \}(= \Sigma).
\]
If we take 

$$k = Z (j u + j u)$$

as the norm of $X$, then (A.1)' and (A.2) are satisfied. Note that (A.3) is also satisfied (we may assume that $G$ is a closed subgroup of $O(N)$).

**Theorem 4.1** Let $\lambda > 0$, $1 < p < q < p^*$, with $p^* = Np/(N - p)$ (for $N > p$); $p^* = \infty$ (for $N \leq p$). Assume that $f \in L^{p/(p-1)}(\Omega)$ is $G$-invariant and $f \leq 0$ a.e. in $\Omega$. Then the variational inequality (14) has a $G$-invariant nontrivial solution.

**Proof.** By assumption, $\Psi + I_K$ is a $G$-invariant functional. So, by virtue of Theorem 3.3, $u$ is a $G$-invariant solution of (14) if it is a critical point of $I := \Psi|_\Sigma + (I_K)|_\Sigma$.

Since $I$ is not $C^1$, in order to construct a nontrivial solution of (14), we can not depend on the classical mountain pass lemma. However, we can apply a mountain pass type theorem due to Szulkin [20, Theorem 3.2]. We first verify that $I$ has the mountain pass structure in the Banach space $\Sigma$, that is,

(i) $I(0) = 0$ and there exists a $\rho > 0$ such that $\inf \{I(u) : \|u\| = \rho\} > 0$;

(ii) there exists an $e \in \Sigma$ such that $\|e\| > \rho$ and $I(e) \leq 0$.

Let $u \in \Sigma \cap K$. Then $\int_\Omega fu \leq 0$ since $f \leq 0$ a.e. in $\Omega$. Therefore, by the Sobolev inequality,

$$\Psi(u) \geq C_1\|u\|^p - \frac{1}{q}|u|^q_{L^q} \geq C_1\|u\|^p - C_2\|u\|^q,$$

where $C_1$ and $C_2$ are constants which are independent of $u$. Hence the condition (i) is satisfied since $q > p$. Fix $u \in (\Sigma \cap K) \setminus \{0\}$. We have, for $t \geq 0$,

$$\Psi(tu) = \frac{t^p}{p} \int (|\nabla u|^p + \lambda|u|^p) - \frac{t^q}{q} \int |u|^q - t \int f u.$$  

Since $1 < p < q$, the condition (ii) is also satisfied.

We next show that $I$ satisfies the following condition $(PS)'$, a generalization of the Palais Smale condition (see [20]).

$(PS)'$ If $(u_n)$ is a sequence such that $I(u_n)$ is bounded and

$$\langle \Psi'(u_n), v - u_n \rangle \geq -\varepsilon_n \|v - u_n\| \quad \forall v \in \Sigma \cap K$$

with $\varepsilon_n \to 0$, then $(u_n)$ possesses a convergent subsequence.

In order to check $(PS)'$, we shall use the following result, which is an analogue of Theorem 1.24 of [24].
Proposition 4.2 If $\Omega$ is compatible with $G$, then the embeddings

$$\Sigma = W^{1,p}_{0,G} (\Omega) \subset L^q (\Omega), \quad p < q < p^*$$

are compact.

Proof. We follow the proof of Theorem 1.24 of [24]. Assume that $u_n \rightharpoonup 0$ weakly in $W^{1,p}_{0,G} (\Omega)$. We are going to show that $u_n \to 0$ strongly in $L^q (\Omega)$.

We first prove that

$$\sup_{y \in \Omega} \int_{B(y,r)} |u_n|^p \to 0 \quad \text{as } n \to \infty. \quad (17)$$

Here, it is understood that $u_n$ is extended to $\mathbb{R}^N$ by zero.

Let $\varepsilon > 0$. It is clear that

$$\int_{B(y,r)} |u_n|^p \leq \sup_n |u_n|_{L^p(\Omega)}^p m(y, r, G) \quad \forall n$$

since $u_n$ is $G$-invariant (note that if $\text{dist}(x, \Omega) > r$, then $\int_{B(y,r)} |u_n|^p = 0$).

Hence, by (15), there exists $R > 0$ such that

$$\sup_{|y| \geq R} \int_{B(y,r)} |u_n|^p \leq \varepsilon \quad \forall n. \quad (18)$$

While the Rellich theorem implies that $u_n \to 0$ strongly in $L^p (B(0, R + r))$. So there exists $n_0 \in \mathbb{N}$ such that

$$\sup_{|y| \leq R} \int_{B(y,r)} |u_n|^p \leq \varepsilon \quad \forall n \geq n_0. \quad (19)$$

Since $\varepsilon > 0$ is arbitrary, the assertion (17) follows from (18) and (19).

We next prove that $u_n \to 0$ strongly in $L^s (\Omega)$, where $s = p(p + N)/N$.

By the Gagliardo-Nirenberg inequality, we have

$$|u_n|_{L^s(B(y,r))} \leq C|u_n|_{L^p(B(y,r))}^{1-\lambda} \|u_n\|_{W^{1,p}(B(y,r))}^\lambda,$$

where $\lambda = N/(N + p)$. Noticing that $s\lambda/p = 1$, we obtain

$$\int_{B(y,r)} |u_n|^s \leq C^s |u_n|_{L^p(B(y,r))}^{(1-\lambda)s} \int_{B(y,r)} (|\nabla u_n|^p + |u_n|^p).$$

Now, covering $\Omega$ by balls with radius $r$, in such a way that each point of $\Omega$ is contained at most $N + 1$ balls, we find

$$\int_{\Omega} |u_n|^s \leq (N + 1)C^s \sup_{y \in \Omega} \left\{ \int_{B(y,r)} |u_n|^p \right\}^{(1-\lambda)s/p} \|u_n\|_{W^{1,p}_{0,G}(\Omega)}.$$

Thus, by (17), we have $u_n \to 0$ strongly in $L^s (\Omega)$.
Since $|u_n|_{L^p}$ and $|u_n|_{L^{p'}}$ are both bounded, by using the interpolation inequality (i.e., Hölder’s inequality), $|u_n|_{L^q} \leq C|u_n|_{L^p}^{1-\theta}|u_n|_{L^{p'}}^\theta$ or $|u_n|_{L^q} \leq C|u_n|_{L^p}^{\theta}|u_n|_{L^{p'}}^{1-\theta}$ for some $\theta \in [0, 1]$, we can see that $\|u_n\|_{L^q} \to 0$ as $n \to \infty$ for $p < q < p'$.

Now let $(u_n)$ be a sequence such that $I(u_n)$ is bounded and satisfies (16). Putting $v = 2u_n$ in (16), we have $\langle \Psi'(u_n), u_n \rangle \geq -\varepsilon_n \|u_n\|$. Since $\varepsilon_n \to 0$ and $\Psi(u_n)$ is bounded, we obtain, for almost all $n$,

$$C_1 + \|u_n\| \geq \Psi(u_n) - \frac{1}{q} \langle \Psi'(u_n), u_n \rangle = \left( \frac{1}{p} - \frac{1}{q} \right) \int_\Omega (|\nabla u_n|^p + \lambda |u_n|^p) + \left( \frac{1}{q} - 1 \right) \int_\Omega f u_n$$

$$\geq C_2 \|u_n\|^p,$$

where $C_1 = \sup \Psi(u_n)$ and $C_2 = (1/p - 1/q) \min\{1, \lambda\}$. Hence $\|u_n\|$ is bounded. So we may assume that $u_n \rightharpoonup u$ weakly in $\Sigma$, $|\nabla u_n|_{L^p} \to a (\geq |\nabla u|_{L^p})$, and $|u_n|_{L^p} \to b (\geq |u|_{L^p})$.

By Proposition 4.2, $u_n \to u$ strongly in $L^q(\Omega)$ and hence $u_n^{q-1} \to u^{q-1}$ strongly in $L^{q/(q-1)}(\Omega)$. It follows from (16) with $v = u$ that

$$-\varepsilon_n \|u - u_n\| \leq \langle \Psi'(u_n), u - u_n \rangle \leq |\nabla u_n|_{L^p}^{p-1} (|\nabla u|_{L^p} - |\nabla u_n|_{L^p}) + \lambda |u_n|_{L^p}^{p-1} (|u|_{L^p} - |u_n|_{L^p})$$

$$+ \int_\Omega (u_n^{q-1} - u^{q-1})(u_n - u) + \int_\Omega f(u_n - u).$$

Letting $n \to \infty$, we get

$$0 \leq a^{p-1} (|\nabla u|_{L^p} - a) + \lambda b^{p-1} (|u|_{L^p} - b).$$

This implies $\lim |\nabla u_n|_{L^p} = |\nabla u|_{L^p}$ and $\lim |u_n|_{L^p} = |u|_{L^p}$. Then, since $L^p$ is uniformly convex, $u_n$ converges to $u$ strongly in $W^{1,p}(\Omega)$. Thus the condition (PS)' is verified. Hence, to complete the proof of Theorem 4.1, it suffices to apply Theorem 3.2 of [20].

Acknowledgment Authors wish to their gratitude to Hidekazu Asakawa for his several helpful comments for Proposition 3.11.

References


