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Tetsuo Arisawa

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Advanced Research Institute
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Waseda University

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Interpretations and Implications of the Negative Binomial Distributions of Multiparticle Productions

Tetsuo Arisawa*

Waseda University, Tokyo 169-8555, Japan

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Abstract

Number of particles produced in high energy experiments is approximated by the negative binomial distributions. Deriving a representation of the distributions from a stochastic equation, conditions for the process to satisfy the distributions are clarified. Based on them, it is proposed that the multiparticle productions consist of "spontaneous" and "induced" productions. The rate of the induced production is proportional to the number of existing particles. The ratio of the two production rates remains constant during the process. The "NBD space" is also defined where number of particles produced in its subspaces follow the negative binomial distributions with different parameters.

Keywords: Negative Binomial distribution, multiparticle production

*arisawa@waseda.jp
In high energy experiments, accelerated particles are collided to reproduce the situations that occurred only in the early universe in nature. Many particles are created under the circumstances. In the experiments a part of the produced particles in selected events are usually analyzed to understand features of elementary particle processes. However, because the multiparticle production process as a whole can not be handled only by simple perturbative calculations, complete understanding of the phenomena has not been achieved.

It is known that multiplicity distributions of particles produced in the experiments are well described by the negative binomial distributions (NBD) [1–4]. Experiments show that it holds as a general aspect of multiparticle production processes, regardless of colliding particles, such as \( ar{p}p \), \( e^+e^- \), \( pN \), or \( NN \), for the wide range of the energies \( \sqrt{s} \).

Models and interpretations of the distributions have been proposed such as branching of QCD partons [5], a Gamma mixing distribution to the Poisson [6], and productions of clans [7]. To comprehend the general features of the multiparticle production processes, new interpretations for the negative binomial distributions are being attempted in this report.

A stochastic equation of the NBD with parameter \( k \) is expressed as

\[
\frac{dP_n}{dt} = \lambda(t) \left\{ -(k + n)P_n + (k + n - 1)P_{n-1} \right\},
\]

for integer \( n \geq 1 \). For \( n = 0 \), the second term of the right hand side is omitted. \( P_n(t) \) represents a probability that an event contains \( n \) particles which have been produced after collisions. The \( t \) is most simply regarded as time, but interpreted differently as a variable of parton branching in references [5, 8–10].

The \( \lambda(t) \) denotes a particle production rate at \( t \). Unlike the references, the production rate here is assumed \( t \)-dependent. At \( t_0 \), the initial \( t \) of the production, the \( \lambda(t) \) becomes nonzero and the multiparticle productions start. The \( \lambda(t) \) returns to zero and the multi particle productions finish before the observation. In addition, different from the references, the representation (1) of the stochastic equation corresponds to an initial condition \( P_n(t_0) = \delta_{n0} \). At the observation, the produced particle number \( n \) is identified as the observed particle multiplicities. The expression (1) is chosen for the later conveniences.

From the equation (1), \( P_n(t) \) is represented with \( P_{n-1}(t) \),

\[
P_n(t) = \int_{t_0}^{t} dt_n \exp\left\{ -\int_{t_n}^{t} dt'_n(k + n)\lambda(t'_n) \right\} (k + n - 1)\lambda(t_n) P_{n-1}(t_n).
\]
By successive substitutions, \( P_n(t) \) is expressed as follows:

\[
P_n(t) = \int_0^t dt_n \exp\{-\int_{t_n}^t dt'_n (k + n) \lambda(t'_n)\} (k + n - 1) \lambda(t_n) \\
\int_0^{t_n} dt_{n-1} \exp\{-\int_{t_{n-1}}^{t_n} dt'_{n-1} (k + n - 1) \lambda(t'_{n-1})\} (k + n - 2) \lambda(t_{n-1}) \\
\ldots \\
\int_0^{t_2} dt_1 \exp\{-\int_{t_1}^{t_2} dt'_1 (k + 1) \lambda(t'_1)\} k \lambda(t_1) \exp\{-\int_{t_0}^{t_1} dt'_0 k \lambda(t'_0)\},
\]

which leads to a NBD formula

\[
P_n(t) = \frac{\Gamma(k + n)}{\Gamma(1 + n) \Gamma(k)} \exp(-k \int_{t_0}^t dt' \lambda(t')) \\
\{1 - \exp(-\int_{t_0}^t dt' \lambda(t'))\}^n.
\]

In the case of \( k = 1 \), the negative binomial distributions turn to the Bose-Einstein (or geometric) distributions. The negative binomial distribution expressed by (3) will be specified by \( \text{NBD}(k, \int_{t_0}^T dt' \lambda(t')) \) in this report.

The equation (2) (and also (1)) is most easily illustrated in a simple particle branching picture, temporarily assuming \( k \) as an initial number of particles at \( t_0 \). After \( t_0 \), particles are produced one by one, the \( i \)-th particle at \( t_i \), where \( i = 1, 2, \ldots, n \) and \( t_0 < t_1 < \ldots < t_n < t \). Probability that the \( i \)-th particle is produced between \( t_i \) and \( t_i + dt_i \) is considered as \((k + i - 1)\lambda(t_i)dt_i\). It is supposed that each of \((k + i - 1)\) particles which exist just before \( t_i \) branches with the same rate of \( \lambda(t_i) \). The factor \( \exp\{-\int_{t_i}^{t_{i+1}} dt'_i (k + i) \lambda(t'_i)\} \) means a probability that \((k + i)\) particles remain without branching from \( t_i \) to \( t_{i+1} \).

In order to satisfy the negative binomial distributions, following conditions on \( \lambda(t) \) and \( k \) are required for the multiparticle productions.

1. \( \lambda(t) \) is common for all existing particles at any \( t \) in each event or collision.

2. \( \int_{t_0}^T dt \lambda(t) \) are the same for all events, where \( T \) denotes the \( t \) at the observations. (It is possible that \( \lambda(t) \) differs event by event as a function.)

3. \( k \) remains constant at any \( t \) in each event.

4. \( k \) is the same for all events.
These consequences of the negative binomial distributions provide general features of multiparticle productions, at least as approximations. Or these requirements provide a basis to discuss limit of application of the negative binomial distributions.

Experiments [1] revealed that \( k \) is not an integer and decreases with \( \sqrt{s} \). These contradict the simple assumption that \( k \) is a number of initial particles or partons. (For example, number of partons in (anti)protons is expected to increase with \( \sqrt{s} \) as observed in jet productions.) In addition, if \( k \) is the initial number of particles, there should exist a fixed number of partons at initial time \( t_0 \) in every event. To avoid this feature, slightly different distributions from the negative binomial distributions were proposed [8, 9, 11] by assuming simple distributions of initial particle numbers. Instead of modifying the NBD, other interpretations for \( k \) will be considered in this report: constant ratios of different production rates.

First, \( k \) is possibly interpreted as a ratio of particle production probabilities from initial particles and from produced particles. It is assumed that the production or the branching rate from the initial particles as a whole is written as \( k \lambda(t) \) at any \( t \). While the \( \lambda(t) \) means the production rate from each particle produced after \( t_0 \). By this interpretation, behaviors of \( k \) with \( \sqrt{s} \) become more understandable. At lower \( \sqrt{s} \), particles produced by branching rarely induce additional productions because energies distributed to them are too low. Then, the productions by initial particles are dominant and \( k \) becomes large. Valence quarks in colliding hadrons or \( q \) and \( \bar{q} \) from \( e^+e^- \) annihilations may be regarded as the initial particles here.

In this context, the initial particles and the produced particles are treated differently with the different production rates. Furthermore, the initial particles are not observed or counted as final particles unlike the produced particles, although the formers continue to exist and repeat branching with the latters until the productions finish with \( \lambda(t) = 0 \). In the equation (1), \( n \) represents the number of the particles produced after \( t_0 \) and observed at \( t \) which results in the negative binomial distributions. Contributions from the initial particles are included in \( k \), but not in the number of particles, \( n \), to be detected.

However, both types of particles should be just QCD partons in terms of QCD theory. Instead of employing different types of particles, the negative binomial distributions are reinterpreted introducing two types of particle productions: "spontaneous" and "induced" productions.
The spontaneous productions occur independently of the other particles. On the other hand, a presence of particles induces an additional particle to be produced as the induced production. The rate of the induced production is proportional to the number of the present particles. The ratio between the spontaneous production rate and the induced production rate (per one existing particle) is assumed constant, and the value is identified as \( k \) of the negative binomial distributions. As seen in the equation (1) the transition rate at \( t \) from the \( n \) particle state to the \( n + 1 \) state is expressed \( (n + k)\lambda(t) \), where \( \lambda(t) \) means the induced production rate for one particle.

In the branching pictures, the spontaneous production represents a production from vacuum. The vacuum may relate to the flows of initial or colliding particles. The induced production corresponds to a branching from existing particles where the branching rate from each particle is supposed equal.

The words "spontaneous" and "induced" are borrowed from the theories of the photon radiation. Einstein proposed the two categories of photon emissions from atoms: spontaneous and induced emissions. Quantum mechanics proved later that the rates of the two types of emissions are always the same \( (k=1) \) [12]. Although these theories are usually applied for the identical Bosons, it is not considered here whether the particles for the multiparticle productions are indistinguishable or not.

As mentioned previously, the negative binomial parameter \( k \) for the multiparticle productions remains constant at any \( t \) in any events. Meanwhile, the ratio of the spontaneous and the induced emissions is always 1 for photons of any frequencies and polarizations that are emitted from any kind of atoms at any temperatures. This comparison may lead to formulate a comparable theory for multiparticle productions where the ratio of the spontaneous and the induced productions is unchanged but not unity. Acting the generation operator \( a^{\dagger} \) which satisfies usual commutation relation for Bosons on a \( n \) particle state \( |n> \), an equation

\[
a^{\dagger} |n> = \sqrt{n + k} |n + 1>
\]

is expected in those theories.

Number of the spontaneous productions in an event follows the Poisson distributions of average \( k \int_{t_0}^{t} dt' \lambda(t') \). Number of the induced productions originated from one spontaneously produced particle obeys the Bose-Einstein distributions, NBD\( (1, \int_{t_0}^{t} dt' \lambda(t')) \)

\[
P_n(t; t_s) = \exp \left( - \int_{t_s}^{t} dt' \lambda(t') \right) \left\{ 1 - \exp \left( - \int_{t_s}^{t} dt' \lambda(t') \right) \right\}^n
\]  \( \quad \text{(4)} \)
where $t_s$ denotes the $t$ of the spontaneous production.

Because $k$ is embedded in the average number of the Poisson distributions of the spontaneous production, reasoning becomes unnecessary for a non-integer and fixed value of particle numbers for each event.

The interpretation of negative binomial distributions are extended using a generating function:

$$G(z) = \left( \frac{\exp\left( - \int_{t_0}^t dt' \lambda(t') \right)}{1 - \{1 - \exp(-\int_{t_0}^t dt' \lambda(t'))\}z} \right)^k.$$  \hspace{1cm} (5)

Suppose that $N_0$ independent regions are created at $t_0$. In the $i$-th region, the ratio of the spontaneous and the induced production rates is assumed $k_i$. $k_i$ can differ for regions and $N_0$ can be different for events, but the sum

$$k = \sum_{i=1}^{N_0} k_i$$

is required to be constant for all events. Also, the integral of the induced production rate $\int_{t_0}^t dt' \lambda(t')$ must be the same for all regions. Then the total multiplicity $n = \sum_{i=1}^{N_0} n_i$, the sum of the multiplicities of each region $n_i$, results in the negative binomial distributions because the total generating function with $k$ becomes a product of generating functions of all regions.

By advancing the arguments with the generating function (5), "NBD space" will be defined. Setting the number of the independent regions $N_0$ to infinity, $N_0 \to \infty$, and introducing a continuous coordinate $r$, the NBD parameter $k$ is expressed as

$$k = \int dr \, \kappa(r)$$

with a density function $\kappa(r)$.

The NBD space is defined as the $r - t$ space where multiparticle productions are attached as follows. Each $r = \text{const}$ line on the plane is regarded as an independent region of particle productions. If a particle is produced spontaneously at a point $(r, t)$, then induced productions or particle branching that follow occur on the same line of $r = \text{const}$. The spontaneous and the induced production rates at a point $(r, t)$ are expressed $\kappa(r)\lambda(t)drdt$ and $\lambda(t)drdt$, respectively.

On a line $r = \text{const}$ where a spontaneous production arises at $t_s$, number of induced particles are distributed according to the Bose-Einstein distributions of (4). Total number
of particles in the whole space obeys the NBD\( (k, \int_0^{t'} dt' \lambda(t')) \). Number of particles produced in any rectangular regions of \([r_1, r_2] \times [t_1, t_2]\) in the \(r - t\) space follow the NBD\( (\int_{r_1}^{r_2} dr \kappa(r), \int_{t_1}^{t_2} dt' \lambda(t')) \). By changing the detection area \([r_1, r_2]\), the NBD parameter \(k\) varies.

Experiments show that the parameter \(k\) changes when the rapidity windows to observe particles are changed \([1, 2]\). So the rapidity may be thought of as the NBD space coordinate \(r\) here. But in particle production processes, it is difficult to assume that all particles branched from the same origin are detected with the same rapidity. Therefore, at least observed rapidities could not be identified as the variable \(r\) above in a strict sense. However, the variation of the parameter \(k\) in regard to the observed rapidity regions implies the necessity of the coordinate such as \(r\) on which a density function, \(\kappa(r)\), are defined.

Because the observed rapidities of particles do not completely reflects a coordinate of the production point \(r\), the NBD fit to the multiplicities are expected to be deteriorated by limiting rapidity regions in experiments.

Summaries. In this report, representations (2) and (3) for the negative binomial distributions are derived by integrating the stochastic equation (1). By interpreting the formulas, conditions on the multiparticle production processes are listed. Then it is proposed that the multiparticle production processes are composed of the spontaneous and the induced productions. The parameter \(k\) of the negative binomial distributions is interpreted as a constant ratio of the two production rates.

Generating function (5) of the negative binomial distributions is used to introduce a continuous coordinate \(r\) on which a density function of \(k\) is defined. Then, the "NBD space" is constructed where particles produced in it’s subspaces result in the negative binomial distributions of various parameters. The understanding of the negative binomial distributions here would contribute to the interpretations of the other NBD phenomena.

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    (Addison-Wesley, Massachusetts, 1966).
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